Expectation

Statistics for Data Science
CSE357 - Fall 2021

Arithmetic Mean

$$X = \left[2, 3, 3, 4, 4, 4, 5, 5, 6 \right]$$

Arithmetic Mean

$$X = \{2, 3, 3, 4, 4, 4, 5, 5, 6\}$$

$$\frac{\text{SUM}}{\text{COUNT}} = \frac{\sum_{i \in \text{range}(X)} x_i}{|X|}$$

An Alternative to Calculate the Mean?

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$$= 2 * \frac{|x = 2|}{|X|} + 3 * \frac{|x = 3|}{|X|} + \ldots$$

$$= \sum_{v \in X} v * \frac{|x = v|}{|X|}$$

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$$\mathbf{E}(X) = \int x dF(x) = \begin{cases} \sum_x x f(x) & \text{if } X \text{ is discrete} \\ \int x f(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

denoted: $\mathbf{E}(X) = \mathbf{E}X = (x) = \mu = \mu x$

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Alternative Conceptualization: If I had to summarize a distribution with only one number, what would do that best?
(the average of a large number of randomly generated numbers from the distribution)

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the PDF, $f(x)$, is the derivative of the CDF, $F(x)$.

Conceptually: Approximate … en the distr … no other information: what value should I expect?

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Examples:

$X \sim$ Bernoulli(p):

$X \sim$ Uniform(-3,1):

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$$f(x) = \binom{n}{x} p^x (1-p)^{(n-x)} = \binom{1}{x} p^x (1-p)^{(1-x)}$$

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(practice)

$$= \frac{a+b}{2}$$

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Variance, Second Moment

Conceptually: The expected difference from the mean.

The ***variance*** of X is:

$$V(X) = \sigma^2 = E(X - \mu)^2 = \int (x - \mu)^2 dF(x)$$

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$$= p(1-p)(p+1-p) = p(1-p)$$

Distribution	Mean	Variance
Point mass at a	a	0
Bernoulli(p)	p	$p(1-p)$
Binomial(n,p)	np	$np(1-p)$
Geometric(p)	$1/p$	$(1-p)/p^2$
Poisson(λ)	λ	λ
Uniform(a,b)	$(a+b)/2$	$(b-a)^2/12$
Normal(μ,σ^2)	μ	σ^2
Exponential(β)	β	β^2
Gamma(α,β)	$\alpha\beta$	$\alpha\beta^2$
Beta(α,β)	$\alpha/(\alpha+\beta)$	$\alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1))$
t_ν	0 (if $\nu > 1$)	$\nu/(\nu-2)$ (if $\nu > 2$)
χ^2_p	p	$2p$
Multinomial(n,p)	np	see below
Multivariate Normal(μ,Σ)	μ	Σ

discrete

continuous

(Wasserman, 2003)

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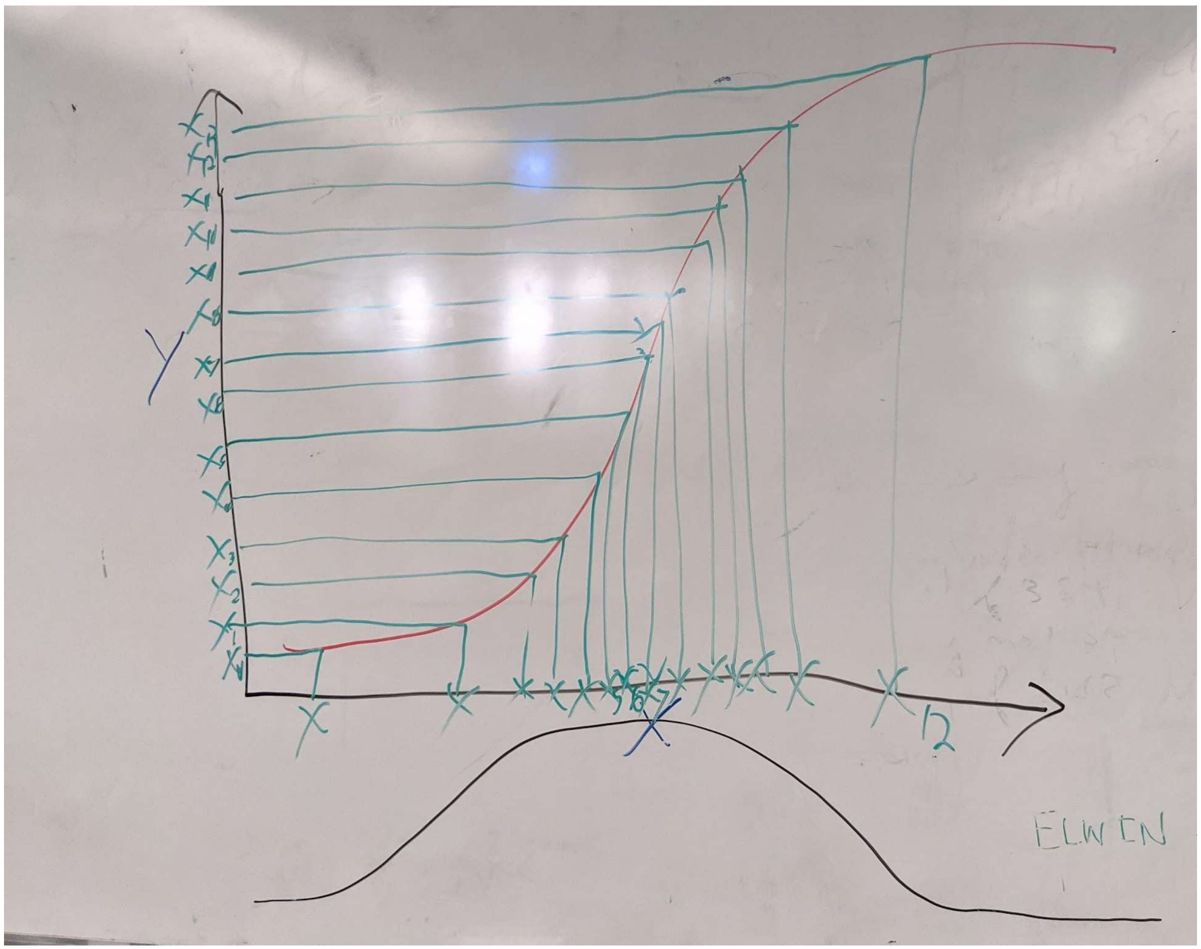
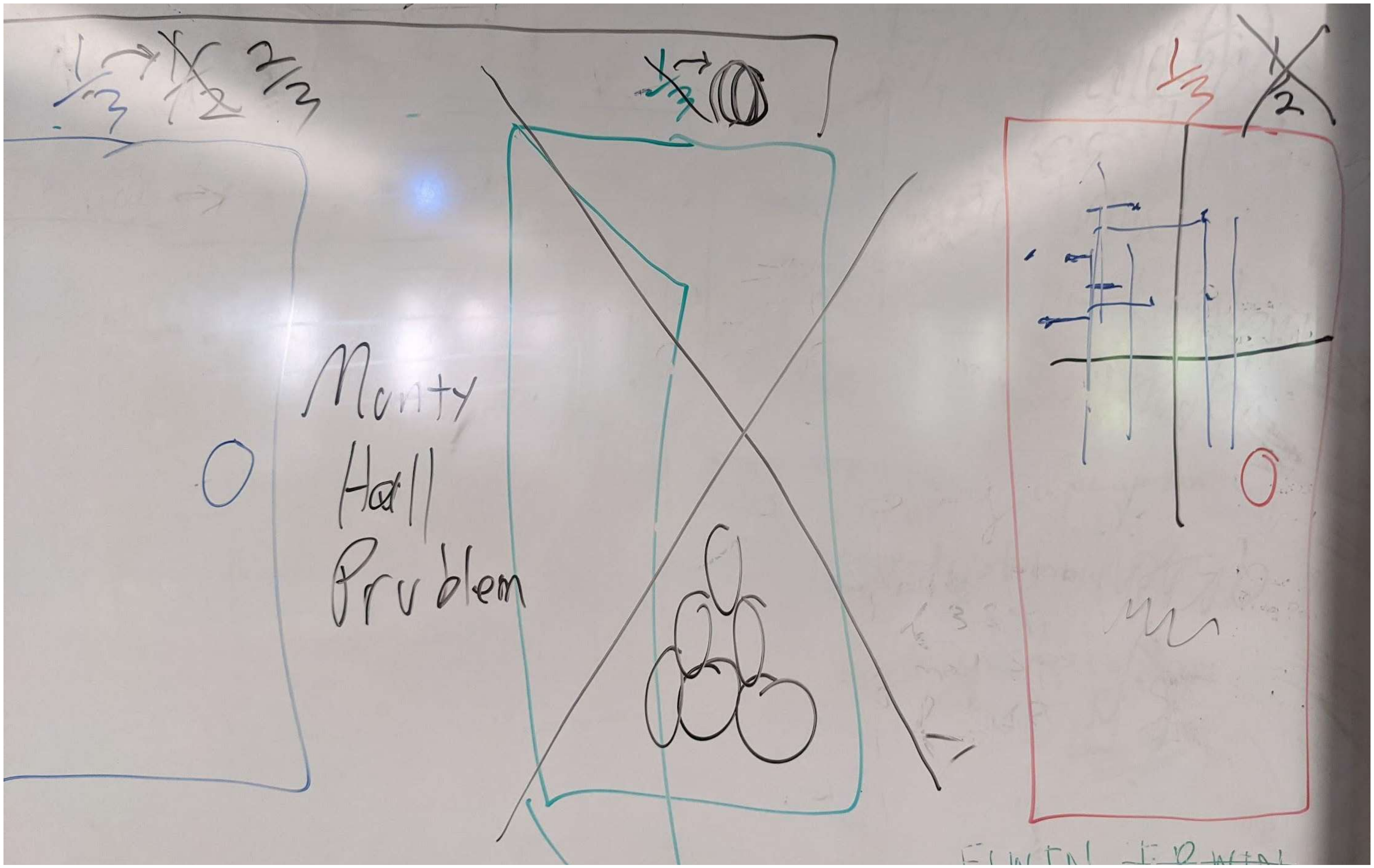
CDF to PDF trick

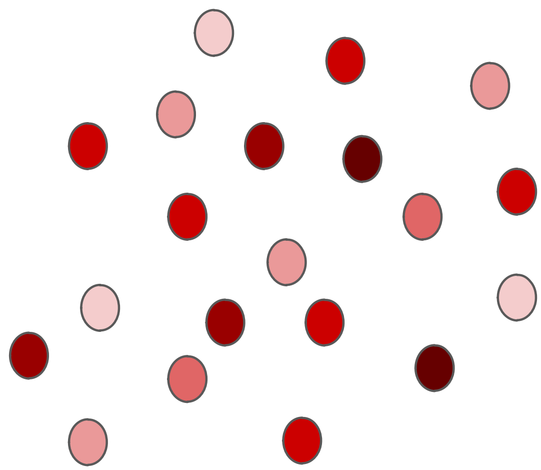


Monty Hall Problem



Population and Samples

Population



Sample

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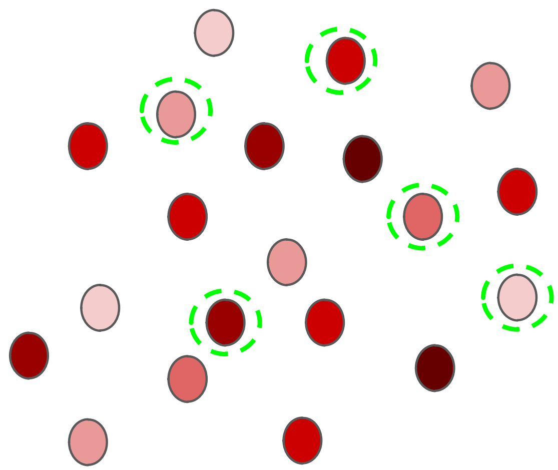

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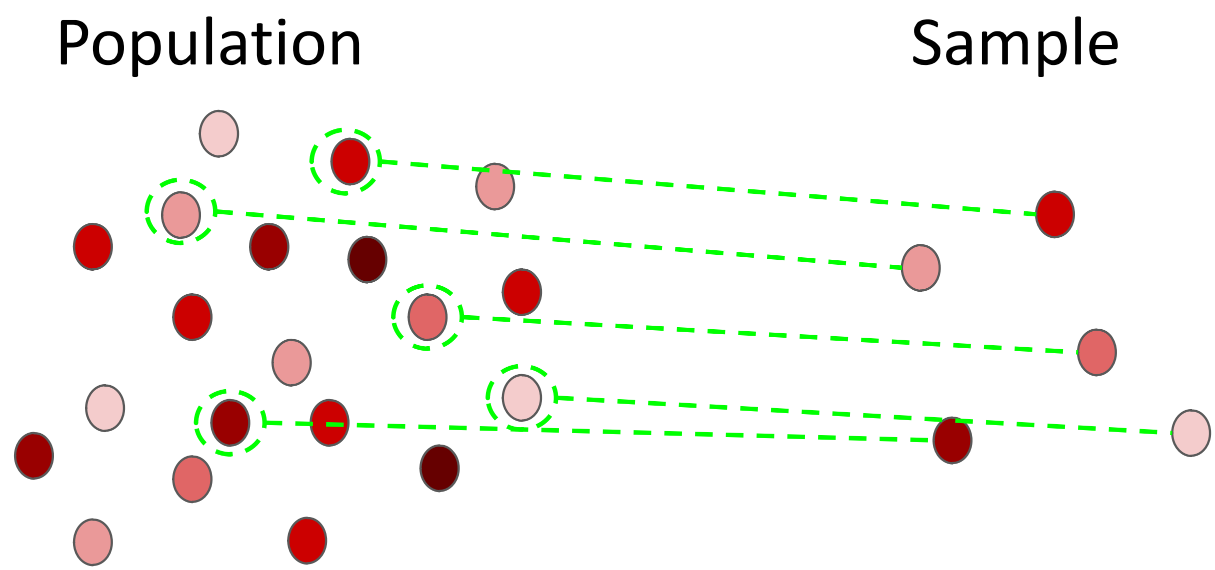

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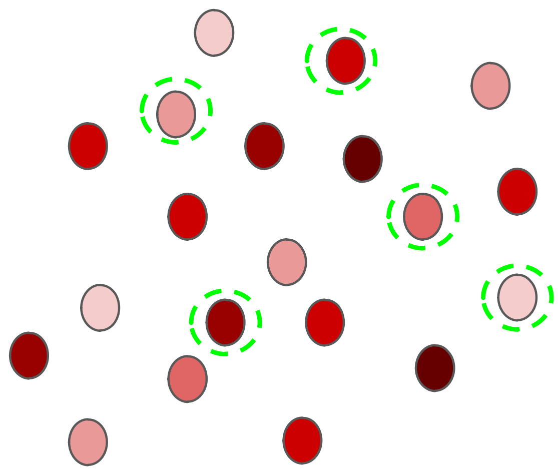

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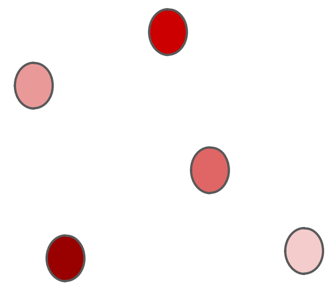
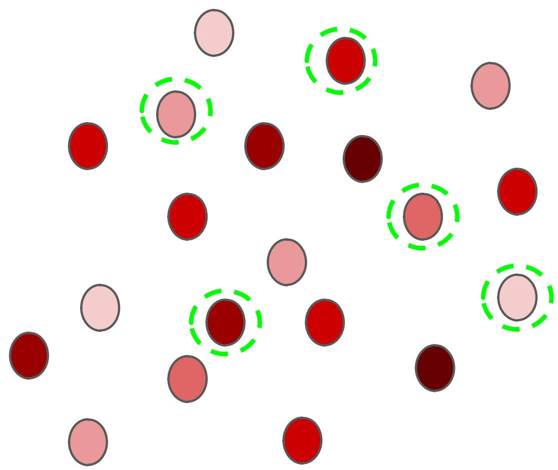


complete: described by pdf of RV
expectation: $E(X)$ or μ
variance: $Var(X)$ or σ^2

presumed random subset of population
sample mean: $\bar{X}$
sample variance: s^2

Population and Samples

Population



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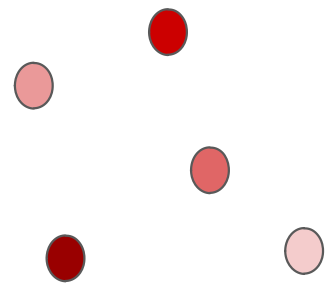

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Population and Samples

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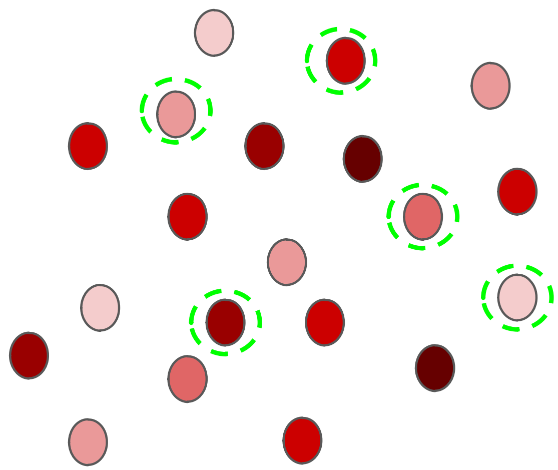

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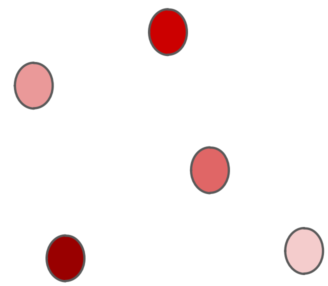

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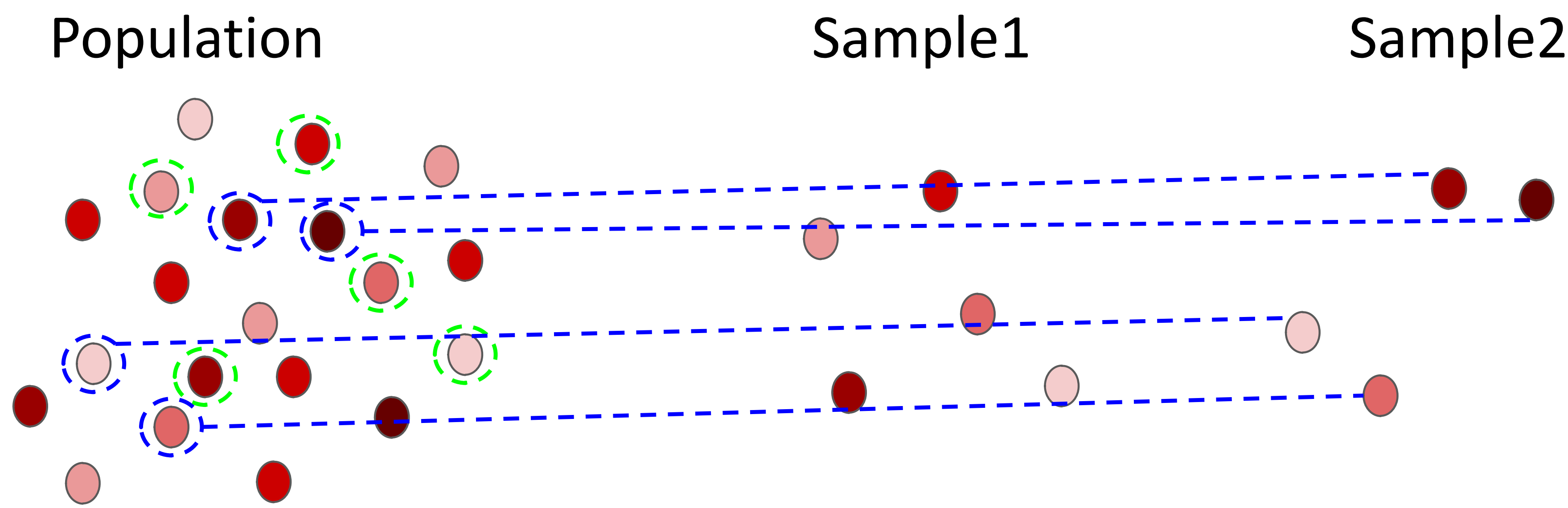
$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$$

Population and Samples



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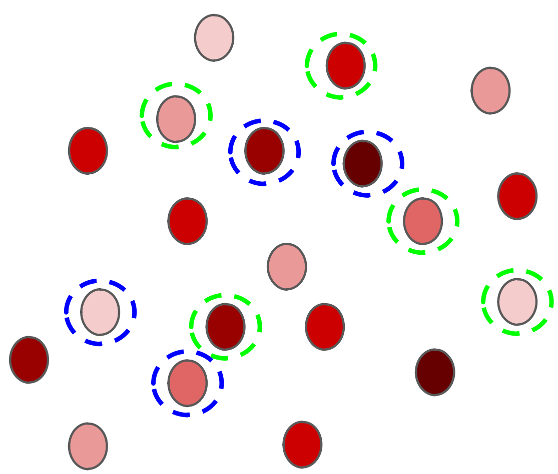
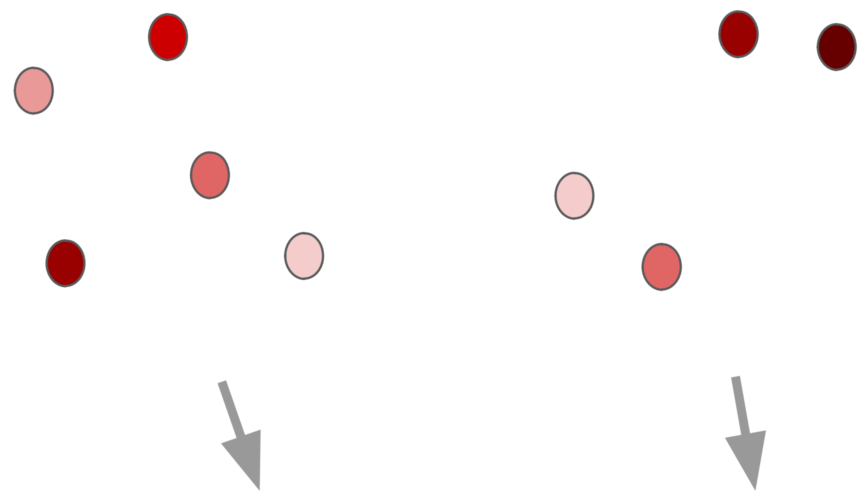


Sample1 **Sample2**



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presumed random subset of population

	Sample1	Sample2
sample mean:	$\bar{X}_1$	$\bar{X}_2$
sample variance:	s_1^2	s_2^2

Law of Large Numbers

"Weak Law of Large Numbers" (WLLN)

$$if\ X_1, \ldots, X_n are \text{ iid } then \bar{X}_n \rightarrow^p \mu$$

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independent and identically distributed

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means $P(|\bar{X} - \mu| > \epsilon) \rightarrow 0$

The sample mean converges with the population mean in probability for every $\epsilon > 0$ ($\bar{X}$ is close to μ with high probability)

Central Limit Theorem

Weak LLN: $if\ X_1, \ldots, X_n are\ \text{iid}\ then \bar{X}_n \rightarrow^p \mu$

Can we describe the distribution of $\bar{X} - \mu$?

$\sqrt{n}(\bar{X}_n - \mu)$ converges to a normal distribution.